SIMULATED POWER CHARACTERISTICS OF TWO MRPP STATISTICS

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Abstract

The symmetric distance function $\triangle_{I,J}$ plays a vital role in the MRPP (multiresponse permutation procedure) for it defines the structure of the underlying analysis space of MRPP. In particular, the form of the symmetric function in consideration is given by

$$\triangle_{I,J} = \left(\sum_{q=1}^{s} |X_{qI}-X_{qJ}|^{p}\right)^{v/p} \text{ hence } p\geq 1 \text{ and}$$

$v>0$ (p is not relevant when $s=1$, the univariate case). For $v>1$, the underlying analysis space of MRPP is nonmetric. The case $p=2$ and $v=1$ corresponds to a Euclidean space. Corresponding to $v=1$ and $v=2$ are two MRPP statistics δ_1 and δ_2, respectively with differing power characteristics. The differential power performance of these two statistics has an important consequence on the geometry of the rejection region of the statistical test. The results of the power comparisons of the data from simulated distributions are presented and the significance of the results are discussed.

Key words: metric space, analysis space, congruence, non-invariance

1. Introduction

Multiresponse permutation procedures (MRPP) introduced and investigated by Mielke and others (Mielke, Berry and Johnson, 1976; O'Reilly and Mielke, 1980; Brockwell, Mielke and Robinson, 1982; Brown, 1982) form a wide class of nonparametric tests to detect difference in observed responses from different groups of objects.

The procedure considers N objects for which s measurements are associated with each object ($s>1$). Let $\Omega = \{w_1,\ldots,w_N\}$ denote the finite population of N objects with w_I corresponding to the Ith object.

Let $x = [X_{1I},\ldots,X_{sI}]$ be the transposed column vector of s measurements object. Let $S_1,\ldots,S_{g+1}$ denote the subgroup of objects ($S_i \in \Omega$, $i=1,\ldots,g+1$) resulting from the application scheme using some a priori basis of classification. The subgroups $S_1,\ldots,S_{g+1}$ represent an exhaustive partitioning of N objects comprising Ω into g well-defined disjoint classes plus an additional disjoint subset, S_{g+1}, consisting of unclassified objects and let

$$n_{g+1} = N - \sum_{i=1}^{g} n_i.$$

The statistic of interest is defined as

$$\delta = \sum_{k=1}^{g} C_k \epsilon_k$$

where ϵ_k denotes the average of the distance measures between objects of the kth subgroup or class given by

$$\epsilon_k = \binom{n_k}{2}^{-1} \sum_{I<J} \triangle_{I,J}\, I_{S_k}(w_I,w_J).$$

The distance measure between objects W_I and W_J is given by

$$\triangle_{I,J} = ||X_I, X_J||$$

The form of $\triangle_{I,J}$ is presently limited to

$$\triangle_{I,J} = \left[\sum_{i=1}^{s} (X_{iI} - X_{iJ})^2 \right]^{v/2}$$

where $v>0$.

$$I_{S_k}(w_I, w_J) = \begin{cases} 1, \text{ if both } w_I \in S_k, \text{ and } w_J \in S_k & I,J = 1,\ldots,N \\ 0, \text{ otherwise.} \end{cases}$$

C_k are positive weighting constants for $k=1,\ldots,g$ and

$$\sum_{k=1}^{g} C_k \doteq 1$$

If $v=1$, then $\triangle_{I,J}$ is the Euclidean distance between w_I and w_J.

2. The symmetric Distance $\triangle_{I,J}$

The symmetric distance function $\triangle_{I,J}$ (which is a special form of a U-Statistic) is a symmetric kernel of degree 2 (Hoeffding, 1948). This function plays a vital role in the MRPP inference technique for it defines the structure of the underlying analysis space of MRPP. The analysis space refers to the space of coordinates for which a particular distance function is used in the subsequent statistical analysis of the data. In particular, the form of the symmetric function currently in consideration is given by

$$\triangle_{I,J} = \left(\sum_{q=1}^{s} |X_{qI} - X_{qJ}|^{P} \right)^{v/P}$$

where $p \geq 1$ and $v>0$ (p is not relevant when $s=1$, the univariate case). For $v_1>1$, the underlying analysis space of MRPP is nonmetric (i.e. the triangle inequality property of a metric space fails). The analysis space of MRPP is a Euclidean space when $p=2$ and $v=1$. It is noted that the usual Euclidean space defines a distance function that is intuitively meaningful to a common experimeter or observer. For this reason, the Euclidean space would be commonly referred to as the data space. While the validity of a permutation test is not affected by these geometric considerations, the rejection region of any test is highly dependent on the underlying geometry. The effect on the power of a permutation test has been demonstrated by two simulation studies (Mielke et al. 1981b and Mielke and Berry, 1982). The results of these studies indicate that the choice of $v=1$ leads to specific advantages over $v=2$ (e.g. superior detection efficiency for locations shifts of bimodal distributions and heavy tailed unimodal distribution).

The results of these two studies are significant in light of the fact that the majority of statistical techniques in current use are based on $v=2$. For example, the permutation version of one-way analysis of variance is characterized by

$$c_k = \frac{(n_k - 1)}{R - g}\;;\quad N=R, g \geq 2, s=1 \;\&\; v=2$$

Let $F = \frac{MS_A}{MS_w}$ be the ordinary one-way analysis of variance statistic. Then the identity relating F and δ is given by

$$N\delta = \frac{2(NB-A^2)}{N-g+(g-1)F}$$

where $A = \Sigma X_I$ and $B = \Sigma X_I^2$. These results are given by Mielke et al. 1982. For g=2, the F-statistic reduces to the two-sided two-sample t test, i.e.

$$F = \frac{MS_A}{MS_w} =$$

$$t^2 = \frac{(\bar{Y}_1-\bar{Y}_2)^2}{S_p^2(1/n_1 + 1/n_2)}$$

Because F and the two-sample t test depend on v=2, the previously mentioned geometry problem of the underlying analysis space is a relevant concern for the permutation version of these commonly used tests.

3. Non-normal Invariance Principle for MRPP

While the two mentioned simulation studies (Mielke et al, 1981b and Mielke and Berry, 1982) revealed some differences on the power of MRPP statistics for values of v=1 and v=2, a theoretical investigation of this kind of comparison is yet to be undertaken because a theory of asymptotic power comparisons involving nonparametric test of this type is presently undeveloped.

The slow development of such a theory could be attributed to the lack of invariance principle when v=1 (i.e. the asymptotic null distribution of an MRPP static depends on the underlying distribution of the response measurements. In fact, Brockwell et. al. (1982) constructed an example of non-invariance (dependence on the parent distribution F) of the asymptotic distribution of δ (centered and scaled) for x,y = |x-y|. These results are in fact hardly encouraging as far as asymptotic statistical inference is concerned.

In lieu of an asymptotic power comparison of the two MRPP statistic δ_1 and δ_2 correspond to v=1 and v=2 respectively, a simulation approach hereby referred to as the matched pair approach is used for computing power estimates. More specifically, the present investigation studies the power characteristics of v=1 and v=2 by simulating the MRPP statistics directly from the data (i.e. no rank transformations are involved). Comparisons are made based on simulation procedures outlined in (Victoria 1986).

Various alternatives of interest have been considered for power comparisons like scale and location alternatives as well as alternatives involving differences in distribution. In reality, the exact form of the observed differences is much more complicated than simply a location on scale shift.

The results of the simulation investigation are given in Tables 1 and 2. Under the location alternative shift, the results show that δ_1 is superior for the u-shaped, Cauchy and the symmetric kappa distribution. Table 2 which uses a different set of starting seed shows that δ_1 performs slightly better than δ_2 for both Laplace and bimodal normal distributions and no obvious difference for the logistic distribution in agreement with Pearson Type III approach of obtaining power estimates (Victoria, 1986). The power characteristics of δ_1 and δ_2 are suprisingly equivalent for the normal case, contrary to usual expectation. δ_1 performs poorly relative to δ_2 for the uniform distribution. For the other alternatives considered no further significant difference is observed except for the u-shaped distributions which show consistent better perfomance of δ_2 relative to δ_1.

4. Conclusion

The issue of power comparisons of MRPP statistics is an important one because it deals with the use of Euclidean distances in statistical methods in place of the squares of such distances. The impact of this geometric consideration on statistical inference proceduce is just beginning to be recognized. This impact was clearly demonstrated through the power comparison of $\delta_1(v=1)$ and $\delta_2(v=2)$ where δ_1 correspond to the MRPP statistic for which the analysis space and data space are congruent loosely speaking and δ_2 is the MRPP statistic for which two spaces are incongruent. The results of the simulation show that the power characteristics of v=1 show definite specific advantages over v=2 (e.g., location shifts involving heavy tailed distributions) while at the same time maintaining power characteristics that are comparable to the power efficiency of v=2 (e.g., location shifts involving heavy tailed distributions) while at the same time maintaining power characteristics that are comparable to the power efficiency of v=2 in instances where the latter offers a slight advantage. In addition v = 1 possesses an intuitive appeal to the non-statistician experiment than do criteria based on sums of squares (v=2).

Ordinarily power comparison results are obtained from theory and verified through simulation. But the difficulty with this approach is that there does not exist a method for obtaining asymptotic power comparison of δ_1 and δ_2. To date there are only two asymptotic results that have been presented namely, the frequent occurence of deviations from normality for the null distributions of MRPP statistics for either small or large finite populations (Mielke, 1979) and the convergence of the null distribution of the MRPP statistics to an infinite sum of independent chi-squares which depend on the underlying distribution of the data (Brochwell et. al, 1982) for $v \neq 2$. Since invariance does not hold, it appears that a theory for asymptotic power comparison of v=1 versus v=2 will be difficult to obtain. At this time, power comparison based on simulation appears to be the only tractable solution.

5. References

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Table 1
Confidence Intervals for Mean Power Difference ($_{\beta}\delta_1-_{\beta}\delta_2$) Using Matched-Pair Method
(First Set of Initial Starting Seeds)

Parent Distribution	α-Level .10	.05
Normal Location shift of = $Q_{.75}-Q_{.50}$	(-0.028±.039)	(-0.033±.035)
Bimodal Normal "	(0.001±.010)	(0.010±.021)
Laplace "	(0.059±.031)	(0.032±.036)
Logistic "	(0.008±.045)	(0.001±.049)
Uniform "	(-0.030±.021)	(-0.038±.031)
Kappa (r=.30) "	(0.098±.030)	(.068±.026)
Cauchy "	(0.148±.038)	(0.084±.017)
Kappa (r=.40) "	(0.107±.024)	(0.067±.036)
Kappa (r=.50) "	(0.078±.035)	(0.064±.043)
U-shaped with 3/2 x $2I_{(-1,1)}(x)$	(0.140±.026)	(0.188±.024)
U-shaped with 3/2 x $2I_{(-1,1)}(x)$ versus 5/2 x $4I_{(-1,1)}(x)$	(0.014±.007)	(0.008±.006)
Chi-Square (d.f. = 4 versus d.f. = 6)	(0.004±.027)	(-0.029±.027)
Exponential (= 1) Scale Shift = .4	(-0.020±.027)	(-0.036±.052)
Exponential (= 1) Scale Shift = .7	(0.008±.030)	(-0.005±.019)

Table 2
Confidence Intervals for Mean Power Difference ($_{\beta}\delta_1-_{\beta}\delta_2$) Using a Different Set of Starting Seeds

Parent Distribution	α-Level .10	.05
Biomodal Normal-Location shift of = $Q_{.75}-Q_{.50}$	(-0.047±.017)	(0.010±0.021)
Normal "	(0.004±0.026)	(0.001±0.22)
Laplace "	(0.026±.021)	(0.025±0.018)
Chi-Square (d.f. = 4 vs. d.f. = 6) using Matched-Pair "	(0.0180±0.026)	(0.014±0.018)
Chi-Square (d.f. = 4 vs. d.f. = 6) using the Three Moment (Pearson) "	(0.012±0.015)	(-0.003±0.024)
Type III "	(0.107±.024)	(0.067±.036)